Chapter 21

Motion in Two Dimensions - Grade 12

21.1 Introduction

In Chapter 3, we studied motion in one dimension and briefly looked at vertical motion. In this chapter we will discuss vertical motion and also look at motion in two dimensions. In Chapter 12, we studied the conservation of momentum and looked at applications in one dimension. In this chapter we will look at momentum in two dimensions.

21.2 Vertical Projectile Motion

In Chapter 4, we studied the motion of objects in free fall and we saw that an object in free fall falls with gravitational acceleration g. Now we can consider the motion of objects that are thrown upwards and then fall back to the Earth. We call this *projectile motion* and we will only consider the situation where the object is thrown straight upwards and then falls straight downwards - this means that there is no horizontal displacement of the object, only a vertical displacement.

21.2.1 Motion in a Gravitational Field

When an object is in a gravitational field, it always accelerates downwards with a constant acceleration g whether the object is moving upward or downward. This is shown in Figure 21.1.

Important: Projectiles moving upwards or downwards always accelerate downwards with a constant acceleration g.

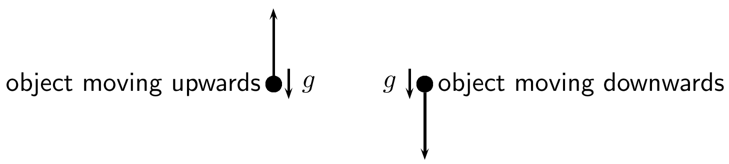


Figure 21.1: Objects moving upwards or downwards, always accelerate downwards.

This means that if an object is moving upwards, it decreases until it stops ($v_f = 0$ m·s^{-1}). This is the maximum height that the object reaches, because after this, the object starts to fall.

Important: Projectiles have zero velocity at their greatest height.

Consider an object thrown upwards from a vertical height h_o. We have seen that the object will travel upwards with decreasing velocity until it stops, at which point it starts falling. The time that it takes for the object to fall down to height h_o is the same as the time taken for the object to reach its maximum height from height h_o.

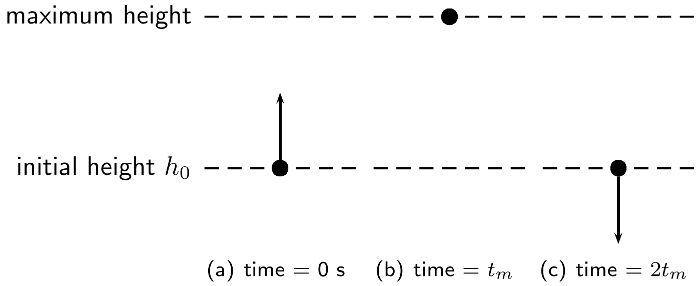


Figure 21.2: (a) An object is thrown upwards from height h_0. (b) After time t_m, the object reaches its maximum height, and starts to fall. (c) After a time $2t_m$ the object returns to height h_0.

Important: Projectiles take the same the time to reach their greatest height from the point of upward launch as the time they take to fall back to the point of launch.

21.2.2 Equations of Motion

The equations of motion that were used in Chapter 4 to describe free fall can be used for projectile motion. These equations are the same as those equations that were derived in Chapter 3, but with $a = g$. We use $g = 9{,}8\,\mathrm{m}\cdot\mathrm{s}^{-2}$ for our calculations.

$$\begin{aligned}
v_i &= \text{initial velocity (m·s}^{-1}\text{) at } t = 0 \text{ s}\\
v_f &= \text{final velocity (m·s}^{-1}\text{) at time } t\\
\Delta x &= \text{height above ground (m)}\\
t &= \text{time (s)}\\
\Delta t &= \text{time interval (s)}\\
g &= \text{acceleration due to gravity (m·s}^{-2}\text{)}
\end{aligned}$$

$$v_f = v_i + gt \tag{21.1}$$

$$\Delta x = \frac{(v_i + v_f)}{2}t \tag{21.2}$$

$$\Delta x = v_i t + \frac{1}{2}gt^2 \tag{21.3}$$

$$v_f^2 = v_i^2 + 2g\Delta x \tag{21.4}$$

Worked Example 132: Projectile motion

Question: A ball is thrown upwards with an initial velocity of 10 m·s^{-1}.

1. Determine the maximum height reached above the thrower's hand.
2. Determine the time it takes the ball to reach its maximum height.

Answer

Step 1 : Identify what is required and what is given
We are required to determine the maximum height reached by the ball and how long it takes to reach this height. We are given the initial velocity $v_i = 10$ m·s^{-1}and the acceleration due to gravity g = 9,8 m·s^{-2}.
Step 2 : Determine how to approach the problem
Choose down as positive. We know that at the maximum height the velocity of the ball is 0 m·s^{-1}. We therefore have the following:

- $v_i = -10\,\mathrm{m \cdot s^{-1}}$ (it is negative because we chose upwards as positive)
- $v_f = 0\,\mathrm{m \cdot s^{-1}}$
- $g = +9{,}8\,\mathrm{m \cdot s^{-2}}$

Step 3 : Identify the appropriate equation to determine the height.
We can use:

$$v_f^2 = v_i^2 + 2g\Delta x$$

to solve for the height.
Step 4 : Substitute the values in and find the height.

$$\begin{aligned} v_f^2 &= v_i^2 + 2g\Delta x \\ (0)^2 &= (-10)^2 + (2)(9{,}8)(\Delta x) \\ -100 &= 19{,}6\Delta x \\ \Delta x &= 5{,}102...m \end{aligned}$$

The value for the displacement will be negative because the displacement is upwards and we have chosen downward as positive (and upward as negative). The height will be a positive number, $h = 5.10m$.
Step 5 : Identify the appropriate equation to determine the time.
We can use:

$$v_f = v_i + gt$$

to solve for the time.
Step 6 : Substitute the values in and find the time.

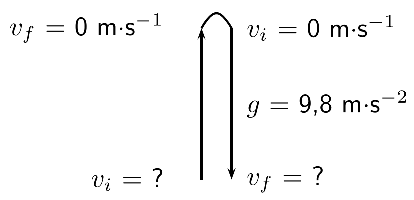
$$\begin{aligned} v_f &= v_i + gt \\ 0 &= -10 + 9{,}8t \\ 10 &= 9{,}8t \\ t &= 1{,}02...s \end{aligned}$$

Step 7 : Write the final answer.
The ball reaches a maximum height of 5,10 m.
The ball takes 1,02 s to reach the top.

Worked Example 133: Height of a projectile

Question: A cricketer hits a cricket ball from the ground so that it goes directly upwards. If the ball takes, 10 s to return to the ground, determine its maximum height.
Answer
Step 1 : Identify what is required and what is given
We need to find how high the ball goes. We know that it takes 10 seconds to go up and down. We do not know what the initial velocity of the ball (v_i) is.
Step 2 : Determine how to approach the problem

A problem like this one can be looked at as if there are two motions. The first is the ball going up with an initial velocity and stopping at the top (final velocity is zero). The second motion is the ball falling, its initial velocity is zero and its final velocity is unknown.



Choose down as positive. We know that at the maximum height, the velocity of the ball is 0 m·s^{-1}. We also know that the ball takes the same time to reach its maximum height as it takes to travel from its maximum height to the ground. This time is half the total time. We therefore have the following for the motion of the ball going down:

- $t = 5\,\mathrm{s}$, half of the total time
- $v_{top} = v_i = 0\,\mathrm{m\cdot s^{-1}}$
- $g = 9{,}8\,\mathrm{m\cdot s^{-2}}$
- $\Delta x = ?$

Step 3 : Find an appropriate equation to use

We are not given the initial velocity of the ball going up and therefore we do not have the final velocity of the ball coming down. We need to choose an equation that does not have v_f in it. We can use the following equation to solve for Δx:

$$\Delta x = v_i t + \frac{1}{2}gt^2$$

Step 4 : Substitute values and find the height.

$$\begin{aligned} \Delta x &= (0)(5) + \frac{1}{2}(9{,}8)(5)^2 \\ \Delta x &= 0 + 122{,}5\mathrm{m} \\ \text{height} &= 122{,}5\mathrm{m} \end{aligned}$$

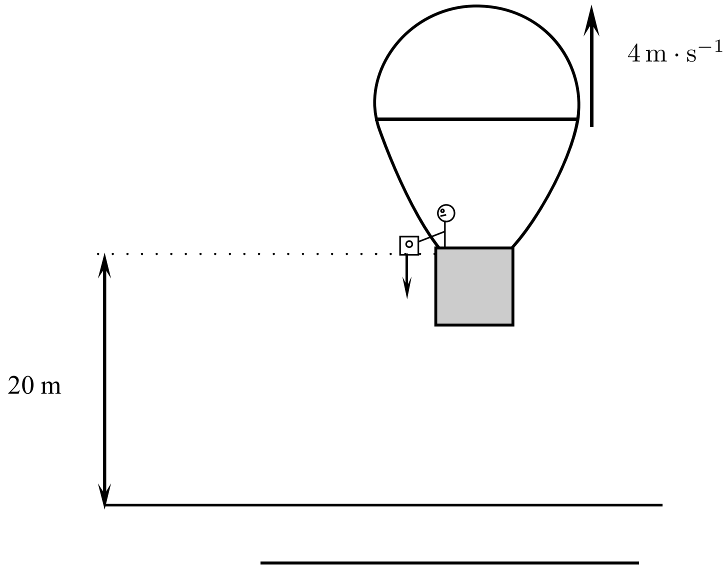
Step 5 : Write the final answer

The ball reaches a maximum height of 122,5 m.

Exercise: Equations of Motion

1. A cricketer hits a cricket ball straight up into the air. The cricket ball has an initial velocity of 20 m·s^{-1}.
   A What height does the ball reach before it stops to fall back to the ground.
   B How long has the ball been in the air for?
2. Zingi throws a tennis ball up into the air. It reaches a height of 80 cm.
   A Determine the initial velocity of the tennis ball.
   B How long does the ball take to reach its maximum height?

3. A tourist takes a trip in a hot air balloon. The hot air balloon is ascending (moving up) at a velocity of 4 m·s^{-1}. He accidentally drops his camera over the side of the balloon's basket, at a height of 20 m. Calculate the velocity with which the camera hits the ground.



21.2.3 Graphs of Vertical Projectile Motion

Vertical projectile motion is similar to motion at constant acceleration. In Chapter 3 you learned about the graphs for motion at constant acceleration. The graphs for vertical projectile motion are therefore identical to the graphs for motion under constant acceleration.
When we draw the graphs for vertical projectile motion, we consider two main situations: an object moving upwards and an object moving downwards.
If we take the upwards direction as positive then for an object moving upwards we get the graphs shown in Figure 21.9.

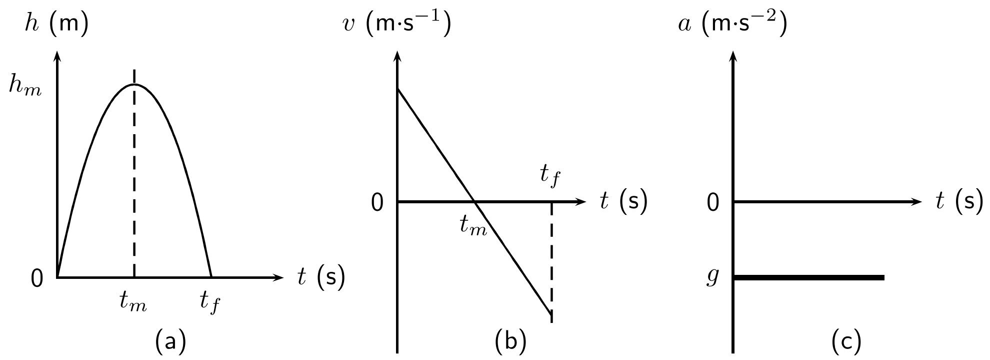


Figure 21.3: Graphs for an object thrown upwards with an initial velocity v_i. The object takes t_m s to reach its maximum height of h_m m after which it falls back to the ground. (a) position vs. time graph (b) velocity vs. time graph (c) acceleration vs. time graph.

Worked Example 134: Drawing Graphs of Projectile Motion

Question: Stanley is standing on the a balcony 20 m above the ground. Stanley tosses up a rubber ball with an initial velocity of 4,9 m·s^{-1}. The ball travels upwards and then falls to the ground. Draw graphs of position vs. time, velocity vs. time and acceleration vs. time. Choose upwards as the positive direction.

Answer

Step 1 : Determine what is required

We are required to draw graphs of

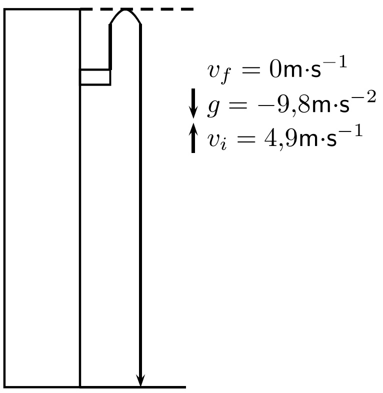
1. Δx vs. t
2. v vs. t
3. a vs. t

Step 2 : Analysis of problem

There are two parts to the motion of the ball:

1. ball travelling upwards from the building
2. ball falling to the ground

We examine each of these parts separately. To be able to draw the graphs, we need to determine the time taken and displacement for each of the motions.



Step 3 : Find the height and the time taken for the first motion.

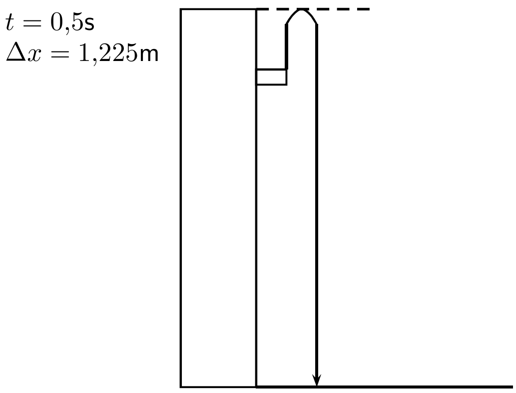
For the first part of the motion we have:

- $v_i = +4{,}9\,\mathrm{m\cdot s^{-1}}$
- $v_f = 0\,\mathrm{m\cdot s^{-1}}$
- $g = -9{,}8\,\mathrm{m\cdot s^{-2}}$

Therefore we can use $v_f^2 = v_i^2 + 2g\Delta x$ to solve for the height and $v_f = v_i + gt$ to solve for the time.

$$\begin{aligned} v_f^2 &= v_i^2 + 2g\Delta x \\ (0)^2 &= (4{,}9)^2 + 2 \times (-9{,}8) \times \Delta x \\ 19{,}6\Delta x &= (4{,}9)^2 \\ \Delta x &= 1{,}225\ m \end{aligned}$$

$$\begin{aligned} v_f &= v_i + gt \\ 0 &= 4{,}9 + (-9{,}8) \times t \\ 9{,}8t &= 4{,}9 \\ t &= 0{,}5\ s \end{aligned}$$



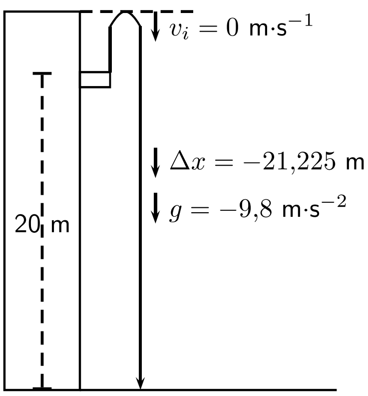
Step 4 : Find the height and the time taken for the second motion.

For the second part of the motion we have:

- $v_i = 0\,\mathrm{m \cdot s^{-1}}$
- $\Delta x = -(20 + 1{,}225)$ m
- $g = -9{,}8\,\mathrm{m \cdot s^{-2}}$

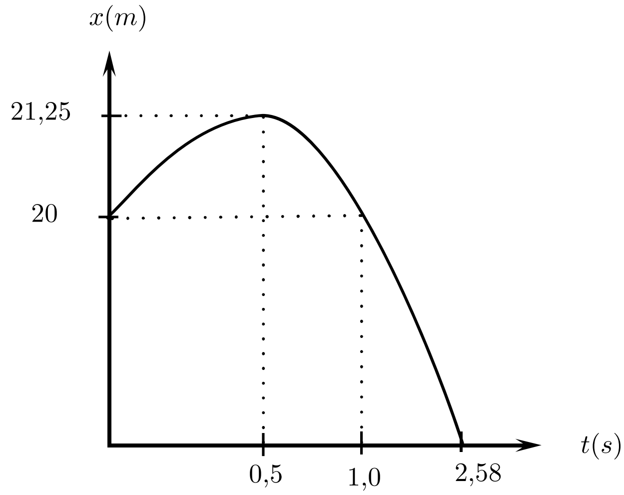
Therefore we can use $\Delta x = v_i t + \frac{1}{2}gt^2$ to solve for the time.

$$\begin{aligned}
\Delta x &= v_i t + \frac{1}{2}gt^2 \\
-(20 + 1{,}225) &= (0) \times t + \frac{1}{2} \times (-9{,}8) \times t^2 \\
-21{,}225 &= 0 - 4{,}9t^2 \\
t^2 &= 4{,}33163... \\
t &= 2{,}08125...\ s
\end{aligned}$$



Step 5 : Graph of position vs. time

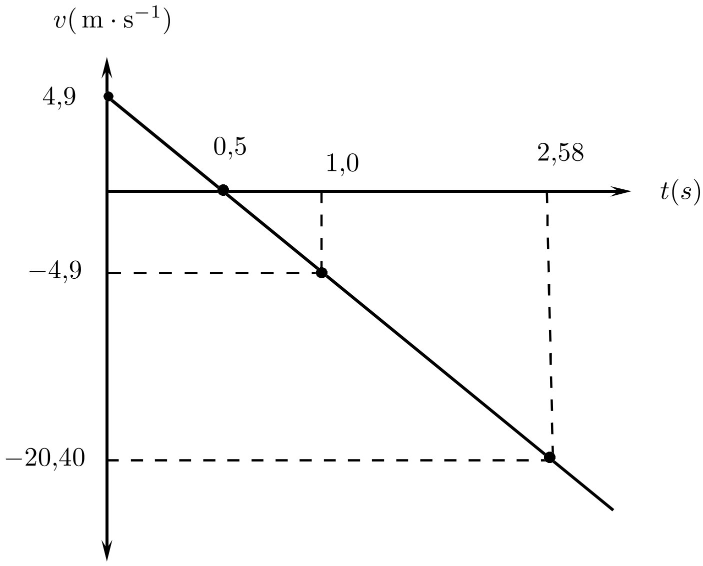
The ball starts from a position of 20 m (at t = 0 s) from the ground and moves upwards until it reaches (20 + 1,225) m (at t = 0,5 s). It then falls back to 20 m (at t = 0,5 + 0,5 = 1,0 s) and then falls to the ground, Δ x = 0 m at (t = 0,5 + 2,08 = 2,58 s).



Step 6 : Graph of velocity vs. time

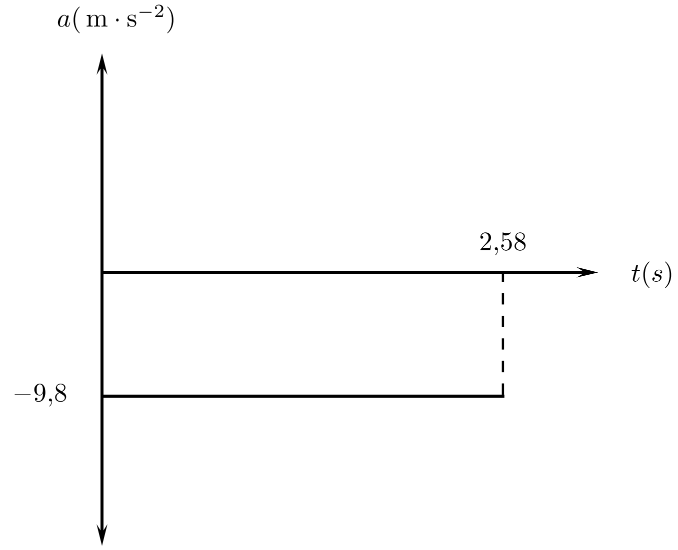
The ball starts off with a velocity of +4,9 m·s^{-1}at t = 0 s, it then reaches a velocity of 0 m·s^{-1}at t = 0,5 s. It stops and falls back to the Earth. At t = 1,0 it has a velocity of -4,9 m·s^{-1}. This is the same as the initial upwards velocity but it is downwards. It carries on at constant acceleration until t = 2,58 s. In other words, the velocity graph will be a straight line. The final velocity of the ball can be calculated as follows:

$$\begin{aligned} v_f &= v_i + gt \\ &= 0 + (-9{,}8)(2{,}08...) \\ &= -20{,}396...\,\mathrm{m \cdot s^{-1}} \end{aligned}$$



Step 7 : Graph of a vs t

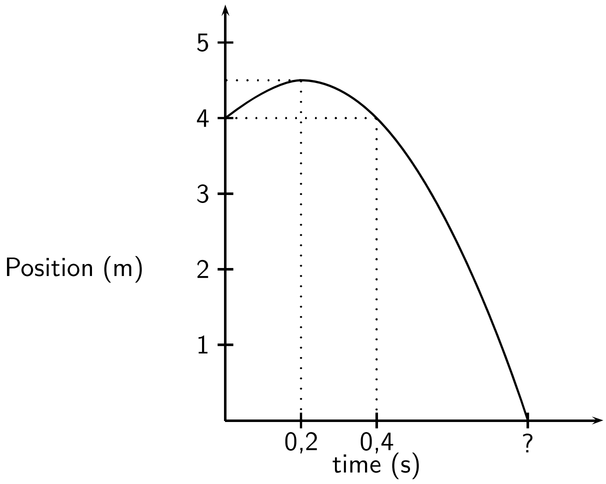
We chose upwards to be positive. The acceleration of the ball is downward. $g = -9.8\,\mathrm{m \cdot s^{-2}}$. Because the acceleration is constant throughout the motion, the graph looks like this:



Worked Example 135: Analysing Graphs of Projectile Motion

Question: The graph below (not drawn to scale) shows the motion of tennis ball that was thrown vertically upwards from an open window some distance from the ground. It takes the ball 0,2 s to reach its highest point before falling back to the ground. Study the graph given and calculate

1. how high the window is above the ground.
2. the time it takes the ball to reach the maximum height.
3. the initial velocity of the ball.
4. the maximum height that the ball reaches.
5. the final velocity of the ball when it reaches the ground.



Answer

Step 1 : Find the height of the window.

The initial position of the ball will tell us how high the window is. From the y-axis on the graph we can see that the ball is 4 m from the ground.

The window is therefore 4 m above the ground.

Step 2 : Find the time taken to reach the maximum height.

The maximum height is where the position-time graph show the maximum position - the top of the curve. This is when t = 0,2 s.

It takes the ball 0,2 seconds to reach the maximum height.

Step 3 : Find the initial velocity (v_i) of the ball.

To find the initial velocity we only look at the first part of the motion of the ball. That is from when the ball is released until it reaches its maximum height. We have the following for this: Choose upwards as positive.

$$\begin{aligned} t &= 0{,}2 \text{ s} \\ g &= -9{,}8 \ \text{m} \cdot \text{s}^{-2} \\ v_f &= 0 \ \text{m} \cdot \text{s}^{-1} (\text{because the ball stops}) \end{aligned}$$

To calculate the initial velocity of the ball (v_i), we use:

$$\begin{aligned} v_f &= v_i + gt \\ 0 &= v_i + (-9{,}8)(0{,}2) \\ v_i &= 1{,}96 \, \text{m} \cdot \text{s}^{-1} \end{aligned}$$

The initial velocity of the ball is 1,96 m·s^{-1}upwards.

Step 4 : Find the maximum height (Δx) of the ball.

To find the maximum height we look at the initial motion of the ball. We have the following:

$$\begin{aligned} t &= 0{,}2 \text{ s} \\ g &= -9{,}8 \ \text{m} \cdot \text{s}^{-2} \\ v_f &= 0 \ \text{m} \cdot \text{s}^{-1} (\text{because the ball stops}) \\ v_i &= +1{,}96 \, \text{m} \cdot \text{s}^{-1} (\text{calculated above}) \end{aligned}$$

To calculate the maximum height (Δx) we use:

$$\begin{aligned} \Delta x &= v_i t + \frac{1}{2} g t^2 \\ \Delta x &= (1{,}96)(0{,}2) + \frac{1}{2}(-9{,}8)(0{,}2)^2 \\ \Delta x &= 0{,}196 \text{m} \end{aligned}$$

The maximum height of the ball is (4 + 0,196) = 4,196 m above the ground.

Step 5 : Find the final velocity (v_f) of the ball.

To find the final velocity of the ball we look at the second part of the motion. For this we have:

$$\begin{aligned} \Delta x &= -4{,}196 \text{ m (because upwards is positive)} \\ g &= -9{,}8 \ \text{m} \cdot \text{s}^{-2} \\ v_i &= 0 \ \text{m} \cdot \text{s}^{-1} \end{aligned}$$

We can use $(v_f)^2 = (v_i)^2 + 2g\Delta x$ to calculate the final velocity of the ball.

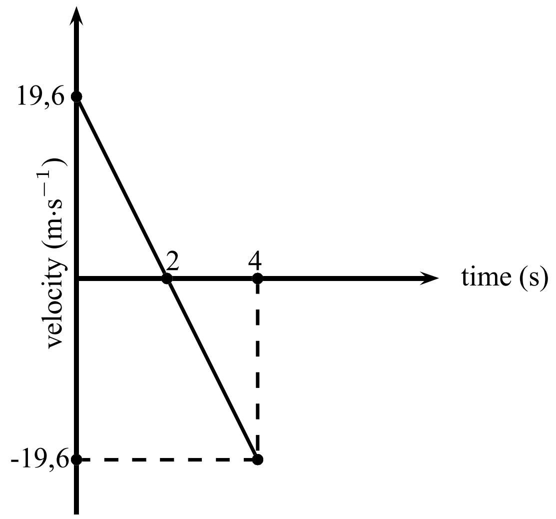
$$\begin{aligned} (v_f)^2 &= (v_i)^2 + 2g\Delta x \\ (v_f)^2 &= (0)^2 + 2(-9{,}8)(-4{,}196) \\ (v_f)^2 &= 82{,}2416 \\ v_f &= 9{,}0687... \, \text{m} \cdot \text{s}^{-1} \end{aligned}$$

The final velocity of the ball is 9,07 m·s^{-1}downwards.

Worked Example 136: Describing Projectile Motion

Question: A cricketer hits a cricket ball from the ground and the following graph of velocity vs. time was drawn. Upwards was taken as positive. Study the graph and answer the following questions:

1. Describe the motion of the ball according to the graph.
2. Draw a sketch graph of the corresponding displacement-time graph. Label the axes.
3. Draw a sketch graph of the corresponding acceleration-time graph. Label the axes.



Answer

Step 1 : Describe the motion of the ball.

We need to study the velocity-time graph to answer this question. We will break the motion of the ball up into two time zones: t = 0 s to t = 2 s and t = 2 s to t = 4 s.

From t = 0 s to t = 2 s the following happens:

The ball starts to move at an initial velocity of 19,6 m·s^{-1}and decreases its velocity to 0 m·s^{-1}at t = 2 s. At t = 2 s the velocity of the ball is 0 m·s^{-1}and therefore it stops.

From t = 2 s to t = 4 s the following happens:

The ball moves from a velocity of 0 m·s^{-1}to 19,6 m·s^{-1}in the opposite direction to the original motion.

If we assume that the ball is hit straight up in the air (and we take upwards as positive), it reaches its maximum height at t = 2 s, stops, turns around and falls back to the Earth to reach the ground at t = 4 s.

Step 2 : Draw the displacement-time graph.

To draw this graph, we need to determine the displacements at t = 2 s and t = 4 s.

At t = 2 s:

The displacement is equal to the area under the graph:

Area under graph = Area of triangle

Area = $\frac{1}{2}$bh

Area = $\frac{1}{2} \times 2 \times 19{,}6$

Displacement = 19,6 m

At t = 4 s:

The displacement is equal to the area under the whole graph (top and bottom). Remember that an area under the time line must be substracted:

Area under graph = Area of triangle 1 + Area of triangle 2

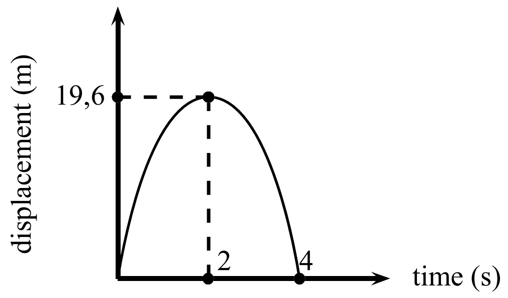
Area = $\frac{1}{2}$bh + $\frac{1}{2}$bh

Area = $(\frac{1}{2} \times 2 \times 19{,}6) + (\frac{1}{2} \times 2 \times (-19{,}6))$

Area = 19,6 - 19,6

Displacement = 0 m

The displacement-time graph for motion at constant acceleration is a curve. The graph will look like this:

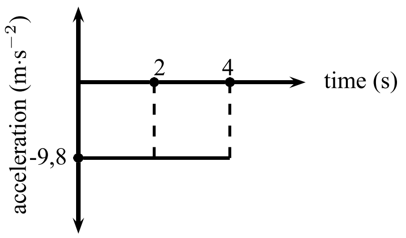


Step 3 : Draw the acceleration-time graph.

To draw the acceleration vs. time graph, we need to know what the acceleration is. The velocity-time graph is a straight line which means that the acceleration is constant. The gradient of the line will give the acceleration.

The line has a negative slope (goes down towards the left) which means that the acceleration has a negative value.

Calculate the gradient of the line:
gradient = $\frac{\Delta v}{t}$
gradient = $\frac{0-19{,}6}{2-0}$
gradient = $\frac{-19{,}6}{2}$
gradient = -9,8
acceleration = 9,8 m·s^{-2}downwards



Exercise: Graphs of Vertical Projectile Motion

1. Amanda throws a tennisball from a height of $1{,}5m$ straight up into the air and then lets it fall to the ground. Draw graphs of Δx vs t; v vs t and a vs t for the motion of the ball. The initial velocity of the tennisball is $2\,\mathrm{m}\cdot\mathrm{s}^{-1}$. Choose upwards as positive.

2. A bullet is shot from a gun. The following graph is drawn. Downwards was chosen as positive

   a Describe the motion of the bullet

   b Draw a displacement - time graph

   c Draw a acceleration - time graph